Synergetic theory for jamming transition in traffic flow

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The theory of a jamming transition is proposed for the homogeneous car-following model within the framework of Lorenz scheme. We represent a jamming transition as a result of the spontaneous deviations of headway and velocity that is caused by the acceleration/braking rate to be higher than the critical value. The stationary values of headway and velocity deviations, and time of acceleration/braking are derived as functions of control parameter (time needed for car to take the characteristic velocity).

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1 Introduction

In recent years considerable study has been given to the traffic problems [1]. It is shown, in particular, that the jamming transition is similar to the conventional gas-liquid phase transition, where the freely moving traffic and the jammed traffic correspond to the gas and liquid phases respectively. The transition between them is caused by growth of car density above a critical value. The congested traffic flow with unstable uniform part leads to the formation of traffic jams where the freely moving traffic and jammed traffic coexists. Within the framework of Ref. [2], the jamming transition

is represented as a first-order phase transition, whose behavior is defined by headway (car density) that acts as the volume (density) and by the inverted delay time (sensitivity parameter) that reduces to temperature.

Our approach is to take into consideration the complete set of freedom degrees as equivalent variables. We obtain the self-consistent analytical description of the jamming transition as result of the self-organization caused by the positive feedback of the headway deviation and acceleration/braking time – on the one hand, as well as the negative feedback of the deviations of headway and velocity – on the other one.

The paper is organized as follows. In Section 2 the self-consistent Lorenz system of the governed equations for the headway and velocity deviations as well as for the acceleration/braking time is obtained. The jamming transition is shown to be supercritical in character (has the second order) if a relaxation time for the first of pointed out quantities does not depend on its value; it transforms to subcritical regime with this dependence appearance. Section 3 deals with the determination of steady-state values for the headway deviation and the acceleration/braking time within adiabatic approximation. Out of the latter limit, the time dependencies for the headway and velocity deviations are studied on the basis of the phase-portrait method. Section 4 contains a short discussion of used assumptions.

2 Basic equations

Within framework of the simplest car-following model the acceleration $\dot{V}$ of a given vehicle as a function of its distance Δx to the front vehicle is defined by equality $\dot{V}=[v_{opt}(\Delta x) - V]/\tau$, where $v_{opt}(\Delta x) = \Delta x/t_0$ is the optimal velocity function (t_0 being a characteristic time interval), $h = Vt_0$ is the optimal headway and τ is the time of acceleration/braking needed for car to take the optimal velocity. It is convenient to introduce deviations $\eta \equiv \Delta x - h$ and $v \equiv \Delta\dot{x} - h/t_0 + V$ of headway Δx and its velocity $\Delta\dot{x}$ from corresponding optimal values h and $h/t_0 - V$. Then, the flow of cars can be described in terms of the pointed out quantities η, v, and τ. The key point of our approach is that the above degrees of freedom are assumed to be of dissipative type, so that, when they are not coupled, their relaxation to the steady state is governed by the Debye-type equations with corresponding relaxation times t_η, t_v, t_τ. Within the simplest approach, equations for the

time dependencies $\eta(t)$, $v(t)$, and $\tau(t)$ are supposed to coincide formally with the Lorenz system that describes the self-organization process [3].

The first of the stated equations has the form

$$\dot{\eta} = -\eta/t_\eta + v, \tag{1}$$

where the dot stands for a derivative with respect to time t. The first term in the right-hand side describes the Debye relaxation during time t_η, the second one is the usual addition. In a stationary state, when $\dot{\eta} = 0$, solution of Eq. (1) defines conventional linear relationship $\eta = t_\eta v$, so that the headway deviation is proportional to the velocity deviation.

The equation for the rate of quantity v variation is supposed to have the nonlinear form

$$\dot{v} = -v/t_v + g_v \eta \tau, \tag{2}$$

where $t_v,\ g_v$ are positive constants. As in Eq. (1), the first term in the right-hand side of Eq. (2) describes the relaxation process of velocity deviation v to the stationary value $v = 0$ determined by a time t_v. The second term describes the positive feedback of the headway deviation η and the time τ of acceleration/braking on the velocity deviation v that results in the increase of value v and, thus, causes the self-organization process.

The kinetic equation for the acceleration/braking time τ

$$\dot{\tau} = (\tau_0 - \tau)/t_\tau - g_\tau \eta v \tag{3}$$

differs from Eqs. (1), (2) as follows: the relaxation of quantity τ occurs not to the zero but to the finite value τ_0, representing the stationary time needed for car to take the characteristic velocity (in other words, τ_0 is the car characteristic); t_τ is a corresponding relaxation time. In Eq. (3) the negative feedback of the quantities η and v on τ is introduced to imply the decrease of acceleration/braking time τ with the growth of the headway and velocity deviations ($g_\tau > 0$ is a corresponding constant).

The equations (1), (2), (3) constitute the basis for self-consistent description of the car-following model driven by the control parameter τ_0. The distinguishing feature of these equations is that nonlinear terms that enter Eqs. (2), (3) are of opposite signs, while Eq. (1) is linear. Physically, the latter means just that the velocity deviation is the derivative of headway deviation with respect to time. The negative sign of the last term in Eq. (3)

can be regarded as a manifestation of Le Chatelier principle, i.e. since an decrease in the acceleration/braking time promote to the formation of a stable car flow, the headway and velocity deviations η and v tend to impede the growth of the acceleration/braking time and, as a consequence, the jamming. The positive feedback of η and τ on v in Eq. (2) plays an important part in the problem. As we will see later, it is precisely the reason behind the self-organization that brings about the traffic jam.

To explain the relaxation transition to the stable jamming state, we will show further that it is quite enough to use the adiabatic approximation: $t_v = 0, t_\tau = 0$. Therefore we could proceed not from Eqs. (2), (3) but from much simple expressions

$$v = a_v \eta \tau, \quad a_v \equiv t_v g_v; \qquad \tau = \tau_0 - a_\tau \eta v, \quad a_\tau \equiv t_\tau g_\tau, \tag{4}$$

which are related to the stationary case $\dot{v} = 0$, $\dot{\tau} = 0$ in Eqs. (2), (3) respectively. The equalities (4) have absolutely clear physical meaning: the increase of the headway deviation η or acceleration/braking time τ leads to growth of the velocity deviation v, whereas the increase of the headway η and velocity v deviations should cause the decrease of acceleration/braking time τ in comparison with characteristic time τ_0 if the car flow is not broken.

After introducing the suitable scales for quantities $\eta,\ v,\ \tau$:

$$\eta_m \equiv (a_v a_\tau)^{-1/2},\ v_m \equiv \eta_m/t_\eta = t_\eta^{-1}(a_v a_\tau)^{-1/2},\ \tau_c \equiv (t_\eta a_v)^{-1}, \tag{5}$$

Eqs. (1), (2), (3) can be rewritten in the simplest form of the well–known Lorenz system:

$$\begin{aligned} \dot{\eta} &= -\eta + v, && (6)\\ \epsilon\, \dot{v} &= -v + \eta\tau, && (7)\\ \delta\, \dot{\tau} &= (\tau_0 - \tau) - \eta v, && (8) \end{aligned}$$

where the relaxation times ratios $\epsilon \equiv t_v/t_\eta$, $\delta \equiv t_\tau/t_\eta$ are introduced and the dot now stands for the derivative with respect to the dimensionless time t/τ_η. In general, the system (6) – (8) can not be solved analytically, but in the simplest case $\epsilon \ll 1$ and $\delta \ll 1$, the left-hand sides of Eqs. (7), (8) can be neglected. Then, the adiabatic approximation can be used to express the velocity deviation v and the acceleration/braking time τ in the form of

the equalities (4). As a result, the dependencies of τ and v on the headway deviation η are given by

$$\tau = \frac{\tau_0}{1+\eta^2}, \qquad v = \frac{\tau_0 \eta}{1+\eta^2}. \tag{9}$$

Note that, under η is in the physically meaningful range between 0 and 1, the acceleration/braking time is a monotonically decreasing function of η, whereas the velocity deviation v increases with η (at $\eta > 1$ we have $\mathrm{d}v/\mathrm{d}\eta < 0$ that has no physical meaning).

Substituting second equality (9) into Eq. (6) yields the Landau–Khalatnikov relation:

$$\dot{\eta} = -\frac{\partial \Phi}{\partial \eta} \tag{10}$$

with the effective potential given by

$$\Phi = \frac{1}{2}\eta^2 - \frac{1}{2}\tau_0 \ln\Big(1+\eta^2\Big). \tag{11}$$

For $\tau_0 < 1$, the η–dependence of Φ is monotonically increasing and the only stationary value of η equals zero, $\eta_e = 0$, so that there is no headway deviations in this case. If the parameter τ_0 exceeds the critical value, $\tau_c = 1$, the effective potential assumes the minimum with non–zero steady state headway deviation $\eta_e = \sqrt{\tau_0 - 1}$ and the acceleration/braking time $\tau_e = 1$.

The above scenario represents supercritical regime of the traffic jam formation and corresponds to the second–order phase transition. The latter can be easily seen from the expansion of the effective potential (11) in power series of $\eta^2 \ll 1$:

$$\Phi \approx \frac{1-\tau_0}{2}\eta^2 + \frac{\tau_0}{4}\eta^4. \tag{12}$$

So the critical exponents are identical to those obtained within the framework of the mean-field theory [4].

The drawback of the outlined approach is that it fails to account for the subcritical regime of the self–organization that is the reason for the appearance of traffic jam and analogous to the first–order phase transition, rather than the second–order one. So one has to modify the above theory by taking the assumption that the effective relaxation time $t_\eta(\eta)$ increases with headway deviation η from initial value $t_\eta/(1+m)$ fixed by a parameter $m > 0$ to

the final one t_η [5]. The simplest two–parameter approximation is as follows:

$$\frac{t_\eta}{t_\eta(\eta)} = 1 + \frac{m}{1 + (\eta/\eta_0)^2}, \tag{13}$$

where $0 < \eta_0 < 1$. The expression for the effective potential (11) then changes by adding the term

$$\Delta\Phi = \frac{m}{2}\eta_0^2 \, \ln\left(1 + \frac{\eta^2}{\eta_0^2}\right) \tag{14}$$

and the stationary values of η are

$$\begin{aligned} \eta_e^m &= \eta_{00}\left\{1 \mp \left[1 + \eta_0^2\eta_{00}^{-4}(\tau_0 - \tau_c)\right]^{1/2}\right\}^{1/2}, \\ 2\eta_{00}^2 &\equiv (\tau_0 - 1) - \tau_c\eta_0^2, \qquad \tau_c \equiv 1 + m. \end{aligned} \tag{15}$$

The upper sign in the right-hand side of Eq. (15) is for the value at the unstable state η^m where the effective potential $\Phi + \Delta\Phi$ has the maximum, the lower one corresponds to the stable state η_e. The corresponding value of the stationary acceleration/braking time

$$\tau^m = \frac{1 + \eta_{00}^2 + \sqrt{\left(1 + \eta_{00}^2\right)^2 - \left(1 - \eta_0^2\right)\tau_0}}{1 - \eta_0^2} \tag{16}$$

smoothly increases from the value

$$\tau_\mathrm{m} = 1 + \eta_0 \sqrt{\frac{m}{1 - \eta_0^2}} \tag{17}$$

at the parameter $\tau_0 = \tau_{c0}$ with

$$\tau_{c0} = \left(1 - \eta_0^2\right)\tau_\mathrm{m}^2 \tag{18}$$

to the marginal value $\tau_c = 1 + m$ at $\tau_0 = \tau_c$.

3 Results

The τ_0–dependencies of η_e, η^m, and τ_e are depicted in Fig. 1. As is shown in Fig. 1a, under the adiabatic condition $t_\tau, t_v \ll t_\eta$ is met and the parameter

τ_0 slowly increases being below τ_c, no traffic jam can form. At the point $\tau_0 = \tau_c$ the stationary headway deviation η_e jumps upward to the value $\sqrt{2}\eta_{00}$ and its further smooth increase is determined by Eq. (15). If the parameter τ_0 then goes downward, the headway deviation η_e continuously decreases up to the point, where $\tau_0 = \tau_{c0}$ and $\eta_e = \eta_{00}$. At this point the headway deviation jump-like goes down to zero. Referring to Fig. 1b, the stationary acceleration/braking time τ_e shows a linear increase from 0 to τ_c with the parameter τ_0 being in the same interval. Then, after the jump down to the value $(1-\eta_0^2)^{-1}$ at $\tau_0 = \tau_c$, the stationary time τ_e smoothly decays to 1 at $\tau_0 \gg \tau_c$. Under the parameter τ_0 then decreases from above τ_c down to τ_{c0} the acceleration/braking time τ_e grows. When the point (18) is reached, the traffic becomes freely moving, so that the stationary acceleration/braking time undergoes the jump from the value (17) up to the one defined by Eq. (18). For $\tau_0 < \tau_{c0}$ again the parameter τ_e does not differ from τ_0. Note that this subcritical regime is realized provided the parameter m, that enters the dispersion law (13), is greater than value

$$m_{\min} = \frac{\eta_0^2}{1-\eta_0^2}. \tag{19}$$

Clearly, according to the picture described, the jamming generation is characterized by the well pronounced hysteresis: the cars initially being at motion with optimal headway between them, begin to deviate only if the acceleration/braking time τ_0 of cars exceeds its limiting value $\tau_c = 1+m$, whereas the acceleration/braking time τ_{c0} needed for uniform car flow is less than τ_c (see Eqs. (17), (18)). This is the case in the limit $t_\tau/t_\eta \to 0$ and the hysteresis loop shrinks with the growth of the adiabaticity parameter $\delta \equiv t_\tau/t_\eta$. In addition to the smallness of δ, the adiabatic approximation implies the ratio $t_v/t_\eta \equiv \epsilon$ is also small. In contrast to the former, the latter does not seem to be realistic for the system under consideration, where, in general, $t_v \approx t_\eta$. So it is of interest to study to what extent the finite value of ϵ could change the results.

Owing to the condition $\delta \ll 1$, Eq. (8) is still algebraic and τ can be expressed in terms of η and v. As a result, we derive the system of two nonlinear differential equations that can be studied by the phase portrait method [5]. The phase portraits for various values of ϵ are displayed in Fig. 2, where the center O represents the stationary state and the saddle point S is related to the maximum of the effective potential. As is seen from the

figure, independently of ϵ, there is the universal section – the "mainstream", that attracts all phase trajectories and its structure is appeared to be almost insensitive to changes in ϵ. Analysis of time dependencies $v(t)$ and $\eta(t)$ reveals that the headway and velocity deviations slow down appreciably on this section in comparison to the rest parts of trajectories that are almost rectilinear (it is not difficult to see that this effect is caused by the smallness of δ). Since the most of time the system is in vicinity of the "mainstream", we arrive at the conclusion that finite values of ϵ do not affect qualitatively the above results obtained in the adiabatic approximation.

4 Discussion

According to the above consideration, the simplest picture of the dissipative dynamic of traffic flow in a homogeneous car-following model can be represented within the framework of Lorenz model, where the headway η and velocity v deviations play a role of an order parameter and its conjugate field, respectively, and the acceleration/braking time τ is a control parameter. The model is examined to show that a jam creates if the car characteristic τ_0 is larger than the critical magnitude τ_c. The above pointed out dissipative regime is inherent in the systems with small values of the relaxation time t_τ for acceleration/braking, being apparently a characteristic of a car-driver, and large ones t_η, t_v for the headway and velocity deviations. According to Ref. [5], in the opposite case $t_\tau \geq t_\eta,\ t_v$, the system behaves in auto-oscillation or stochastic manners.

It is worth-while to note that the above synergetic scheme allows us to explain the collective phenomena of jamming transition in the N-body problem with $N \to \infty$. Then, the question arises: why exactly three variables (the headway and velocity deviations η, v, and acceleration/braking time τ) permit to explain the nontrivial behaviour of N-body problem? The answer to this question gives the theorem by Ruelle and Takens: nontrivial collective behaviour of many-body system (type of the strange attractor) can be represented only in the case, when number of variables is not less than three [3]. The interpretation of this fact is the simplest: the first of the freedom degrees can be chosen as the way along the phase trajectory, the second one corresponds to the negative Lyapunov exponent, ensuring an attraction to this trajectory, the third one acts in opposite manner to give repulsion. In

our case of the self-organization process, the second v and third τ freedom degrees provide the positive and negative feedbacks in Eqs. (2), (3).

The last question to our approach is why the just Lorenz scheme allows us to describes main peculiarities of the jamming transition? The answer is that this is the simplest approach, permitting to catch on the self-organization effects, just as the Landau phenomenological theory of phase transitions describes the great variety of thermodynamical transformations in the simplest way [6]. Let us note in this connection that the effective potential given by the sum of equalities (11), (14) acts the part of the Landau free energy. But the above stated synergetic scheme has the principle difference from the Landau-type theory [2] because the former takes into account a feedback of thermostat (the velocity deviation and the acceleration/braking time) with subsystem under consideration (the headway deviation), whereas the latter does not.

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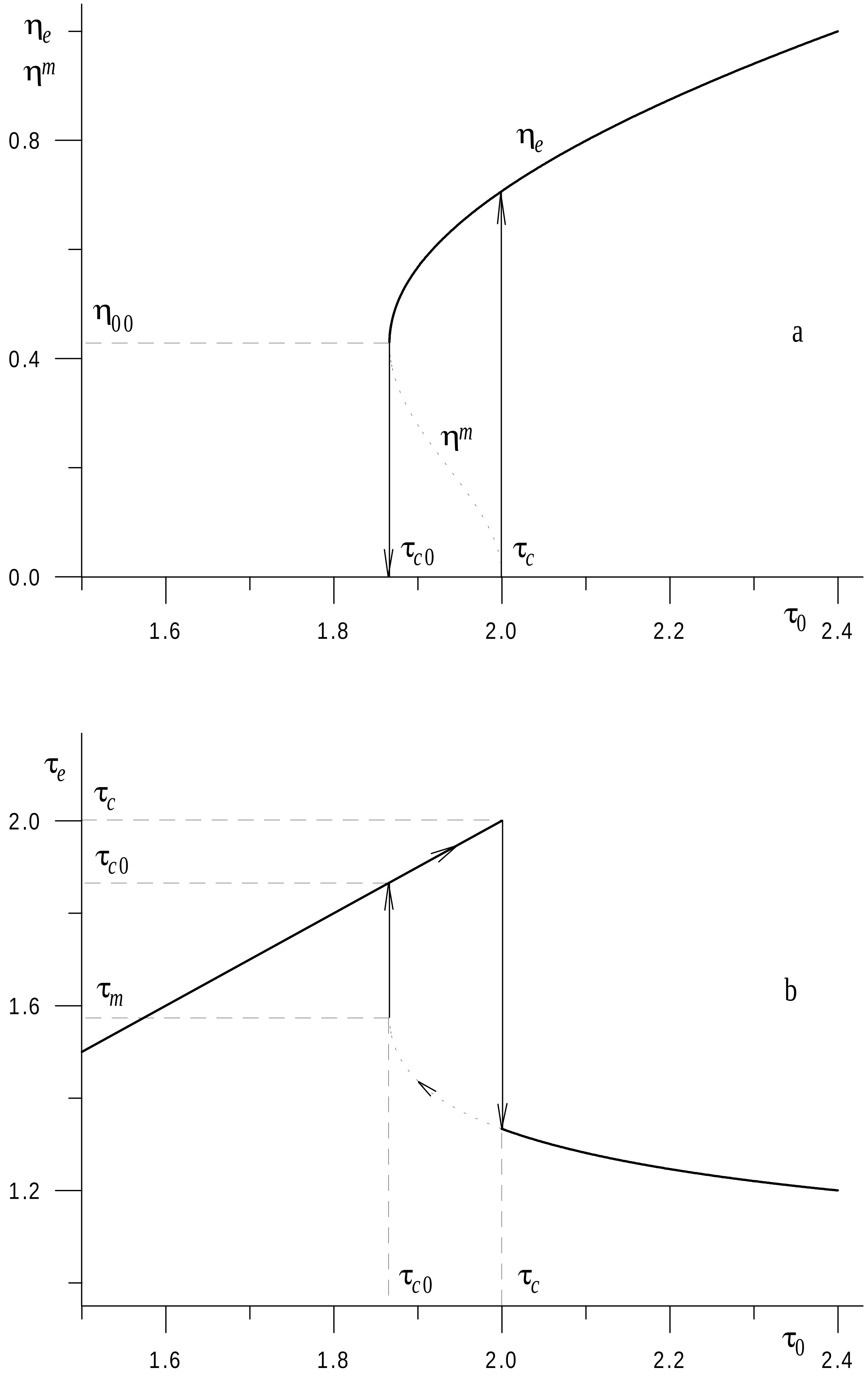
CAPTIONS

Fig. 1. The τ_0–dependencies of the stationary values of: (a) headway deviations η_e, η^m; (b) acceleration/braking time τ_e. The arrows indicate the hysteresis loop.

Fig. 2. Phase portraits in the $\eta - v$ plane at $m = 1,\ \eta_0 = 0.1,\ \tau_0 = 1.25\tau_c$ for: (a) $\epsilon = 10^{-2}$; (b) $\epsilon = 1$; (c) $\epsilon = 10^2$.



ηe
ηm
0.8
ηe
η00
a
0.4
ηm
τc0
τc
0.0
1.6
1.8
2.0
2.2
τ0
2.4
τe
τc
2.0
τc0
τm
b
1.6
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τc0
τc
1.6
1.8
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τ0
2.4

FIG. 1

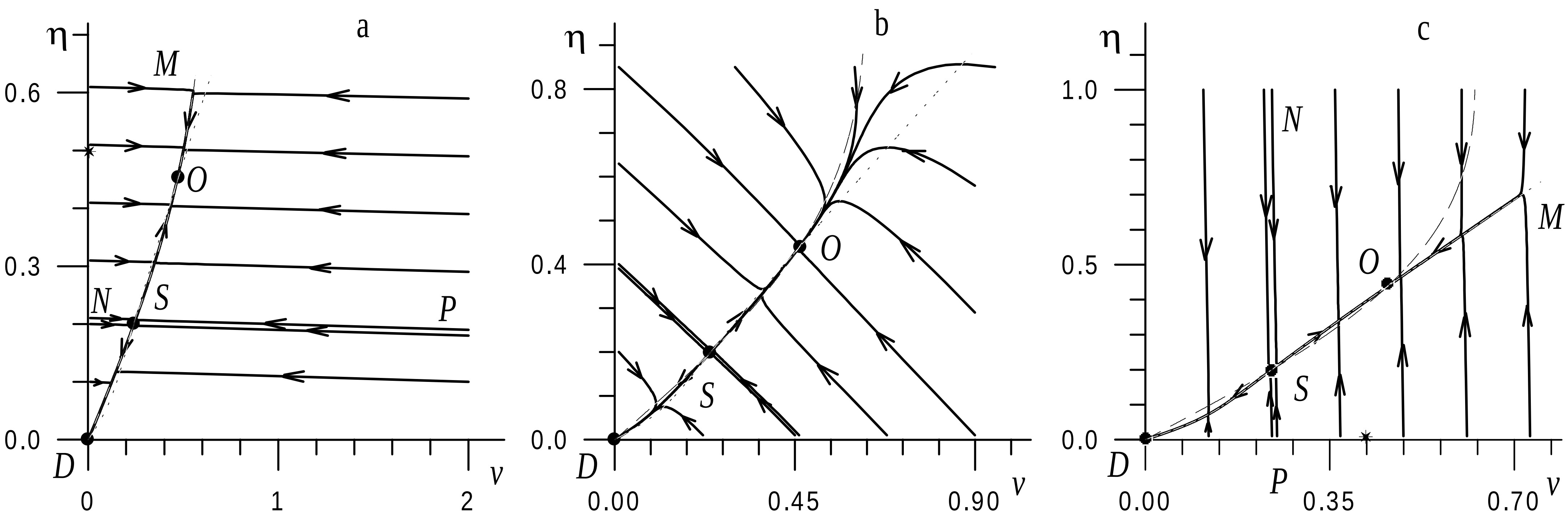
a
η
M
0.6
O
0.3
N
S
P
0.0
D
0
1
2
v
b
η
0.8
O
0.4
S
0.0
D
0.00
0.45
0.90
v
c
η
1.0
N
M
0.5
O
S
0.0
D
P
0.00
0.35
0.70
v

FIG. 2